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Non-constant positive steady states of a prey–predator system with cross-diffusions ☆

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Abstract

In this paper, we study a strongly coupled elliptic system arising from a Lotka–Volterra prey–predator system, where cross-diffusions are included in such a way that the prey runs away from the predator and the predator moves away from a large group of preys. We establish the existence and non-existence of its non-constant positive solutions. Our results show that if $m_1 b < a < 2m_1 b/(1 - m_1 m_2)$ when $0 < m_1 m_2 < 1$ or $a > m_1 b$ when $m_1 m_2 \geqslant 1$, $0 < d_1 < (m_1\tilde{v} - \tilde{u})/\mu_1$, $d_2 > 0$, $d_3 \geqslant 0$ and $d_4 > 1/(m_1\tilde{v} - \tilde{u})$, then there exists (d_1, d_2, d_3, d_4) such that the stationary problem admits non-constant positive solutions. Otherwise, the stationary problem has no non-constant positive solution. In particular, the results indicate that its non-constant positive solutions are mainly created by the cross-diffusion d_4.

Keywords: Prey–predator system; Cross-diffusion; Non-constant positive steady-states; Degree theory

1. Introduction

In this paper, we study non-constant positive steady-state solutions of the following Lotka–Volterra prey–predator system with cross-diffusion effects:

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$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta\big[u(1+d_3v)\big] = r_1\left(u - \dfrac{u^2}{k_1}\right) - m_1uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} - d_2\Delta\big[v(1+d_4u)\big] = r_2\left(v - \dfrac{v^2}{k_2}\right) + m_2uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \not\equiv 0, \quad v(x,0) = v_0(x) \geqslant 0, \not\equiv 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$, and ν is the outward unit normal on $\partial\Omega$; d_i, r_i, k_i, m_i, $i = 1, 2$, are positive constants, and d_j, $j = 3, 4$, are non-negative constants. In the system (1.1), u and v, respectively, represent the population densities of prey and predator species which are interacting and migrating in the same habitat Ω. r_i, $i = 1, 2$, are their intrinsic growth rates, and k_i, $i = 1, 2$, represent the coefficients of the carrying capacity. The terms m_1u and m_2u, respectively, account for the functional response and the conversion rate of the prey captured by the predator. The homogeneous Neumann boundary condition indicates that the system (1.1) is self-contained with zero population flux across the boundary. In diffusion terms, d_i, $i = 1, 2$, represent the natural dispersive force of movement of an individual, while d_1d_3 and d_2d_4 describe the mutual interferences between individuals. d_1d_3 and d_2d_4 are usually referred as cross-diffusion pressures. The system (1.1) means that, in addition to the dispersive force, the diffusion also depends on population pressure from other species. The first cross-diffusion pressure d_3 (or d_1d_3) means the tendency that the prey keeps away from the predator. In a certain kind of prey–predator relationships, a great number of prey species form a huge group to protect themselves from the attack of predator. So d_4 (or d_2d_4) represents the tendency of predators to move away from a large group of preys. See also [8,9] and references therein for the biological background.

In mathematical ecology, the classical prey–predator system, due independently to Lotka and Volterra in the 1920s, reflects only population changes due to predation in a situation where predator and prey densities are not spatially dependent. It does not take into account either the fact that population is usually not homogeneously distributed, nor the fact that predators and preys naturally develop strategies for survival. Both of these considerations involve diffusion processes which can be quite intricate as different concentration levels of predators and preys cause different population movements. Such movements can be determined by the concentration of the same species (diffusion) and that of other species (cross-diffusion).

In [8,9], authors give a more detailed study of a Lotka–Volterra prey–predator system with homogeneous Dirichlet boundary condition and with cross-diffusion effects. They not only establish the existence of positive steady-state solutions, but also carefully analyze the stability and S-shaped bifurcation of positive steady-state solutions.

For ecological models with homogeneous Neumann boundary condition, what is of interest is whether the various species co-exist, in particular, the various species do in non-constant time-independent positive solutions. For example, Pang and Wang [13] first investigate a three species predator–prey model with homogeneous Neumann boundary condition and with cross-diffusion, and demonstrate that cross-diffusion terms can create non-constant positive steady states. The readers also see [1,3,7,10,13,16–19] and references therein for models with homogeneous Neumann boundary condition.

However, as far as we are aware, many authors deal with such a class of ecological models that, in addition to the preys considered, there does not exist other natural source of food to a kind of the predators at least. Namely, there $-\bar{r}_2v$ takes the place of the term $r_2(v - \frac{v^2}{k_2})$ of system (1.1), where $\bar{r}_2$ is the death rate of predators. Authors of [10] investigate an ecological competition model.

For a class of prey–predator models having other natural sources of food to all predators, many works deal with these models with homogeneous Dirichlet boundary condition (see [2,5,8, 9,14,15] and references therein), while a few papers deal with these models with homogeneous Neumann boundary condition. We wonder whether these two kinds of boundary conditions can be adopted by the same model. To our knowledge, for the class of prey–predator models with homogeneous Neumann boundary condition, if we only take into account such a way of movement that the predator chases the prey, then it is easy to check that the corresponding steady-state models only have constant positive solutions, which are not the mainly studying purpose to people. Therefore, it is necessary to take into account another way of movement of the predator as the system (1.1) (see also [8,9]).

In the system (1.1), the predator v diffuses with flux

$$J = -\nabla(d_2 v + d_2 d_4 u v) = -d_2 d_4 v \nabla u - (d_2 + d_2 d_4 u)\nabla v.$$

We observe that, as $-d_2 d_4 v < 0$, the part $-d_2 d_4 v \nabla u$ of the flux is directed toward the decreasing population density of the prey. In particular, when d_4 is large enough, $-d_2 d_4 v \nabla u$ represents that the predator moves away from a large group of preys.

We denote $a = r_1$ and $b = r_2$. By scaling

$$\bar{u} = au, \quad \bar{v} = bv, \quad \bar{d}_3 = \frac{d_3}{b}, \quad \bar{d}_4 = \frac{d_4}{a}, \quad \bar{m}_1 = \frac{m_1}{b}, \quad \bar{m}_2 = \frac{m_2}{a},$$

and for the simplicity of writing, we may drop the '$^-$' sign above and assume that $k_1 = k_2 = 1$, then system (1.1) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta\big[u(1 + d_3 v)\big] = au - u^2 - m_1 uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} - d_2 \Delta\big[v(1 + d_4 u)\big] = bv - v^2 + m_2 uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = au_0(x), \quad v(x,0) = bv_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Since the main purpose of the present paper is to investigate the effects of the cross-diffusion pressures on the existence of non-constant positive steady-state solutions of problem (1.2), we will concentrate on the following strongly coupled elliptic system:

$$\begin{cases} -d_1 \Delta\big[u(1 + d_3 v)\big] = au - u^2 - m_1 uv, & x \in \Omega, \\ -d_2 \Delta\big[v(1 + d_4 u)\big] = bv - v^2 + m_2 uv, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

It is obvious that, when $a > m_1 b$, system (1.3) has the only positive constant solution denoted by $(\tilde{u}, \tilde{v})^T$. Furthermore, if a satisfies: (1) $m_1 b < a < \frac{2m_1 b}{1 - m_1 m_2}$ when $0 < m_1 m_2 < 1$, or (2) $a > m_1 b$ when $m_1 m_2 \geqslant 1$, then $\tilde{u} < m_1 \tilde{v}$ (see Lemma 3.4).

We are mainly interested in non-constant positive solutions of system (1.3). It is said that $(u, v)^T$ is a positive solution of system (1.3) if $u > 0$ and $v > 0$ in $\overline{\Omega}$. We will demonstrate that the cross-diffusion d_4 may be helpful to create more patterns when d_4 is large, in comparison with the results of system (1.3) when d_4 is small (see Theorems 6.1–6.3 and Lemma 5.2).

The contents of the present paper are as follows: In Section 2, we discuss the *Turing instability*. In Section 3, we establish a priori upper and lower bounds for positive solutions of system (1.3). In Section 4, we analyze the linearized steady state problem of system (1.3) at $(\tilde{u}, \tilde{v})^T$. In Section 5, we discuss the non-existence of non-constant positive solutions of system (1.3). In the last section, we study the existence of non-constant positive solutions of system (1.3) for suitable values of the cross-diffusion coefficient d_4 and the diffusion coefficient d_2, respectively, and discuss the bifurcation of non-constant positive solutions of system (1.3) with respect to d_1.

2. Turing instability

For the simplicity, we denote $\mathbf{w} = (u, v)^T$,

$$\mathbf{\Phi}(\mathbf{w}) = \big(d_1 u(1 + d_3 v), d_2 v(1 + d_4 u)\big)^T,$$
$$\mathbf{G}(\mathbf{w}) = \big(au - u^2 - m_1 uv, bv - v^2 + m_2 uv\big)^T,$$

then problem (1.2) can be written as

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} = \Delta \mathbf{\Phi}(\mathbf{w}) + \mathbf{G}(\mathbf{w}), & x \in \Omega,\ t > 0, \\ \dfrac{\partial \mathbf{w}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ \mathbf{w}(x, 0) = \big(au_0(x), bv_0(x)\big)^T, & x \in \Omega. \end{cases} \tag{2.1}$$

Whereas the corresponding spatially homogeneous counterpart of problem (2.1) is

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{G}(\mathbf{w}), \quad t > 0. \tag{2.2}$$

We know that, when $a > m_1 b$, $\mathbf{G}(\mathbf{w}) = 0$ has four non-negative constant solutions: $(0,0)^T$, $(a,0)^T$, $(0,b)^T$ and $(\frac{a-m_1 b}{1+m_1 m_2}, \frac{b+m_2 a}{1+m_1 m_2})^T$ which are denoted by $\mathbf{0}$, $\mathbf{u}^*$, $\mathbf{v}^*$ and $\tilde{\mathbf{w}}$, respectively. Whereas when $a \leqslant m_1 b$, $\mathbf{G}(\mathbf{w}) = 0$ only has three non-negative constant solutions: $\mathbf{0}$, $\mathbf{u}^*$ and $\mathbf{v}^*$.

We now discuss the stability of every non-negative constant equilibrium of problem (2.2).

When $a > m_1 b$, the linearization of problem (2.2) at $\tilde{\mathbf{w}}$ is

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\mathbf{w}, \quad \text{where } \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) = \begin{pmatrix} -\tilde{u} & -m_1 \tilde{u} \\ m_2 \tilde{v} & -\tilde{v} \end{pmatrix}. \tag{2.3}$$

Its character polynomial of $\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})$ is

$$\lambda^2 + (\tilde{u} + \tilde{v})\lambda + \tilde{u}\tilde{v}(1 + m_1 m_2) = 0.$$

It is obvious that $\mathrm{Re}(\lambda) < 0$. Therefore, $\tilde{\mathbf{w}}$ is stable.

By a similar argument, we know that both $\mathbf{0}$ and $\mathbf{u}^*$ are unstable, and $\mathbf{v}^*$ is unstable when $a > m_1 b$, whereas $\mathbf{v}^*$ is stable when $a \leqslant m_1 b$.

Next, we discuss the stability of every non-negative constant equilibrium of the PDE dynamics (2.1).

When $a > m_1 b$, the *Turing instability* refers to “diffusion driven instability,” i.e., the stability of the constant equilibrium $\tilde{\mathbf{w}}$ changing from stable, for the ODE dynamics (2.2), to unstable, for the PDE dynamics (2.1). Here we perform some calculations as that in [17] to find a criterion for the *Turing instability*.

The linearization of system (2.1) at $\tilde{\mathbf{w}}$ is

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} = \mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\Delta\mathbf{w} + \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\mathbf{w}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \mathbf{w}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.4}$$

Let $\{\mu, \varphi(x)\}$ be an eigenpair of $-\Delta$ in Ω subject to homogeneous Neumann boundary condition. Then problem (2.4) has a non-trivial solution of the form $\mathbf{w} = \mathbf{c}\varphi e^{\lambda t}$, where $\mathbf{c} \in \mathbf{R}^2$ is a constant vector, if and only if $(\lambda, \mathbf{c})$ is an eigenpair for the matrix $-\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})$. Hence, the equilibrium $\tilde{\mathbf{w}}$ of system (2.1) is unstable if the matrix $-\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})$ has an eigenvalue with positive real part.

By the direct computations, we have

$$\begin{aligned} &\lambda^2 + \big[d_1(1 + d_3\tilde{v})\mu + \tilde{u} + d_2(1 + d_4\tilde{u})\mu + \tilde{v}\big]\lambda + \det\big[-\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\big] = 0, \\ &\det\big[-\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\big] = A\mu^2 + B\mu + C, \end{aligned}$$

where

$$\begin{aligned} &A = d_1 d_2(1 + d_3\tilde{v} + d_4\tilde{u}) > 0, \\ &B = d_1(1 + d_3\tilde{v})\tilde{v} + d_2\tilde{u}\big[1 + d_4(\tilde{u} - m_1\tilde{v})\big] + d_1 d_3 m_2 \tilde{u}\tilde{v}, \\ &C = \tilde{u}\tilde{v} + m_1 m_2 \tilde{u}\tilde{v} > 0. \end{aligned}$$

Suppose that a satisfies:

(1) $m_1 b < a < \frac{2m_1 b}{1 - m_1 m_2}$ when $0 < m_1 m_2 < 1$, or (2) $a > m_1 b$ when $m_1 m_2 \geqslant 1$, such that $\tilde{u} < m_1\tilde{v}$ (see Lemma 3.4).

Let d_1, d_3 and d_2 (or d_4) be fixed and satisfy that d_1 is a positive number small enough, $d_3 \geqslant 0$ and $d_2 > 0$ (or $d_4 > \frac{1}{m_1\tilde{v} - \tilde{u}}$), then there exists a positive constant D_4 (or D_2) large enough such that, when $d_4 > D_4$ (or $d_2 > D_2$), $\det[-\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})] < 0$ for some μ (see (4.12), (4.13) and Lemma 4.3). Therefore, we have

Lemma 2.1.

(i) *Suppose that a satisfies $m_1 b < a < \frac{2m_1 b}{1 - m_1 m_2}$ when $0 < m_1 m_2 < 1$ or $a > m_1 b$ when $m_1 m_2 \geqslant 1$. Let d_1, d_3 and d_2 (or d_4) be fixed and satisfy that d_1 is a positive number small enough, $d_3 \geqslant 0$ and $d_2 > 0$ (or $d_4 > \frac{1}{m_1\tilde{v} - \tilde{u}}$). Then there exists a positive constant D_4 (or D_2) large enough such that, when $d_4 > D_4$ (or $d_2 > D_2$), the equilibrium $\tilde{\mathbf{w}}$ of* (2.1) *is unstable in Ω, whereas when $d_4 \leqslant D_4$ (or $d_2 \leqslant D_2$), the equilibrium $\tilde{\mathbf{w}}$ of system* (2.1) *is stable in Ω.*

(ii) *For $0 < m_1 m_2 < 1$, if a satisfies $a \geqslant \frac{2m_1 b}{1 - m_1 m_2}$, then the equilibrium $\tilde{\mathbf{w}}$ of system* (2.1) *is stable in Ω.*

By a similar argument, we have

Lemma 2.2.

(i) *Both $\mathbf{0}$ and $\mathbf{u}^*$ of system* (2.1) *are unstable in Ω.*

(ii) *The equilibrium $\mathbf{v}^*$ of system* (2.1) *is unstable when $a > m_1 b$, whereas $\mathbf{v}^*$ is stable when $a \leqslant m_1 b$.*

To compare the stabilities of the non-negative constant equilibria to the PDE dynamics (2.1) with those to the ODE dynamics (2.2), we know that if $a > m_1 b$, the stability of the constant equilibrium $\tilde{\mathbf{w}}$ may change from stable, for the ODE dynamics (2.2), to unstable, for the PDE dynamics (2.1), whereas those of other constant equilibria are invariant.

3. Some prior estimates of positive solutions of (1.3)

In the following, the generic positive constants C, C_i and C_i^*, $i = 1, 2, 3, 4$, etc., will depend only on Ω, a, b, m_1, m_2 and some other positive constants given, but not on $\mathbf{d}$, where $\mathbf{d}$ denotes (d_1, d_2, d_3, d_4).

The main purpose of this section is to give prior upper and lower positive bounds for positive solutions of problem (1.3). To this end, we first cite two known results. Lemma 3.1 is due to Lou and Ni [10], and Lemma 3.2 to Lin, Ni and Takagi [11].

Lemma 3.1 *(Maximum principle). Let $g(x, w) \in C(\Omega \times \mathbf{R}^1)$ and $b_j(x) \in C(\overline{\Omega})$, $j = 1, 2, \dots, N$.*

- (i) *If $w(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta w(x) + \sum_{j=1}^N b_j(x) w_{x_j} + g(x, w(x)) \geqslant 0$ in Ω, $\frac{\partial w}{\partial \nu} \leqslant 0$ on $\partial\Omega$, and $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geqslant 0$.*
- (ii) *If $w(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta w(x) + \sum_{j=1}^N b_j(x) w_{x_j} + g(x, w(x)) \leqslant 0$ in Ω, $\frac{\partial w}{\partial \nu} \geqslant 0$ on $\partial\Omega$, and $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leqslant 0$.*

Lemma 3.2 *(Harnack inequality). Let $c(x) \in C(\overline{\Omega})$, and $w(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x) + c(x) w(x) = 0$ in Ω subject to homogeneous Neumann boundary condition. Then there exists a positive constant $C = C(N, \Omega, \|c(x)\|_\infty)$ such that $\max_{\overline{\Omega}} w(x) \leqslant C \min_{\overline{\Omega}} w(x)$.*

Remark 3.1. Referring to [6,10,13], we know that $C = C(N, \Omega, \|c(x)\|_\infty)$ is monotone increasing with respect to $\|c(x)\|_\infty$. Therefore, if there is a sequence of $\{c_i(x)\}$ satisfying $\|c_i(x)\|_\infty \leqslant C_0$, then there exists $C = C(N, \Omega, C_0)$ such that positive solutions $w_i(x)$ to $\Delta w(x) + c_i(x) w(x) = 0$ in Ω subject to homogeneous Neumann boundary condition satisfy

$$\max_{\overline{\Omega}} w_i(x) \leqslant C(N, \Omega, C_0) \min_{\overline{\Omega}} w_i(x)$$

for all i.

Lemma 3.3. *Let D_1, D_2, D_3 and D_4 be positive constants given. Then there exists a positive constant $C = C(a, b, m_1, m_2, D_1, D_2, D_3, D_4)$ such that, when $d_1 \geqslant D_1$, $d_2 \geqslant D_2$, $d_3 \leqslant D_3$ and $d_4 \leqslant D_4$, the positive solution $(u, v)^T$ of problem (1.3) satisfies*

$$\max_{\overline{\Omega}} u(x) < C, \quad \max_{\overline{\Omega}} v(x) < C, \quad \min_{\overline{\Omega}} v(x) > C^{-1}.$$

Furthermore, if $a \neq m_1 b$, then $\min_{\overline{\Omega}} u(x) > C^{-1}$.

Proof. Set $\phi = u(1 + d_3 v)$ and $\psi = v(1 + d_4 u)$, then problem (1.3) becomes

$$\begin{cases} -d_1 \Delta\phi = au - u^2 - m_1 uv, & x \in \Omega, \\ -d_2 \Delta\psi = bv - v^2 + m_2 uv, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let $x_0 \in \overline{\Omega}$ be a point such that $\phi(x_0) = \max_{\overline{\Omega}} \phi(x)$. Applying Lemma 3.1 to the first equation of problem (3.1), we have $u(x_0) \leqslant a$ and $v(x_0) \leqslant \frac{a}{m_1}$. Thus

$$\max_{\overline{\Omega}} u \leqslant \max_{\overline{\Omega}} \phi(x) = \big(1 + d_3 v(x_0)\big)u(x_0) \leqslant \left(1 + D_3 \frac{a}{m_1}\right) a \triangleq C_1.$$

Similarly, let $x_1 \in \overline{\Omega}$ be a point such that $\psi(x_1) = \max_{\overline{\Omega}} \psi(x)$. Applying Lemma 3.1 to the second equation of problem (3.1), we have $v(x_1) \leqslant b + m_2 u(x_1) \leqslant b + m_2 C_1$. Thus

$$\max_{\overline{\Omega}} v \leqslant \max_{\overline{\Omega}} \psi(x) = v(x_1)\big(1 + d_4 u(x_1)\big) \leqslant (b + m_2 C_1)(1 + D_4 C_1) \triangleq C_2.$$

On the other hand, problem (1.3) can also be written as

$$\begin{cases} -\Delta\phi = \dfrac{a - u - m_1 v}{d_1(1 + d_3 v)}\phi, & x \in \Omega, \\ -\Delta\psi = \dfrac{b - v + m_2 u}{d_2(1 + d_4 u)}\psi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

As $\|\frac{a-u-m_1 v}{d_1(1+d_3 v)}\|_\infty \leqslant \frac{a+C_1+m_1 C_2}{D_1}$ for all $d_1 \geqslant D_1$, Lemma 3.2 holds for ϕ, i.e., $\max_{\overline{\Omega}} \phi \leqslant C_1^* \min_{\overline{\Omega}} \phi$ for some positive constant C_1^*. Therefore,

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \leqslant \frac{\max_{\overline{\Omega}} \phi}{\min_{\overline{\Omega}} \phi} \frac{1 + d_3 \max_{\overline{\Omega}} v}{1 + d_3 \min_{\overline{\Omega}} v} \leqslant C_1^*\Big(1 + d_3 \max_{\overline{\Omega}} v\Big) \leqslant C_1^*(1 + D_3 C_2) \triangleq C_3^*. \tag{3.3}$$

Similarly, as $\|\frac{b-v+m_2 u}{d_2(1+d_4 u)}\|_\infty \leqslant \frac{b+m_2 C_1+C_2}{D_2}$ for all $d_2 \geqslant D_2$, Lemma 3.2 holds for ψ, i.e., $\max_{\overline{\Omega}} \psi \leqslant C_2^* \min_{\overline{\Omega}} \psi$ for some positive constant C_2^*. Hence,

$$\frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \leqslant \frac{\max_{\overline{\Omega}} \psi}{\min_{\overline{\Omega}} \psi} \frac{1 + d_4 \max_{\overline{\Omega}} u}{1 + d_4 \min_{\overline{\Omega}} u} \leqslant C_2^* \frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \leqslant C_2^* C_3^* \triangleq C_4^*. \tag{3.4}$$

By integrating the second equation of problem (1.3) in Ω, we have $\int_\Omega v(b - v + m_2 u)\,dx = 0$. Thus, there exists $x_2 \in \Omega$ such that $b - v(x_2) + m_2 u(x_2) = 0$, which implies that $v(x_2) \geqslant b$. This, combined with (3.4), yields that

$$\min_{\overline{\Omega}} v \geqslant \frac{\max_{\overline{\Omega}} v}{C_4^*} \geqslant \frac{v(x_2)}{C_4^*} \geqslant \frac{b}{C_4^*}.$$

Turning now to prove that $\min_{\overline{\Omega}} u(x) > C^{-1}$ if $a \neq m_1 b$. Here, we still use problem (1.3), but its corresponding variables u, v, diffusion coefficients d_1, d_2, and cross-diffusion coefficients d_3, d_4 etc., are with sub-index base i.

Assume, on the contrary, that there exists a sequence $\{(d_{1i}, d_{2i}, d_{3i}, d_{4i})\}$, $i = 1, 2, 3, \ldots,$ satisfying $d_{1i} \geqslant D_1$, $d_{2i} \geqslant D_2$, $d_{3i} \leqslant D_3$ and $d_{4i} \leqslant D_4$, such that the corresponding positive solutions $(u_i, v_i)^T$ of problem (1.3) with $(d_1, d_2, d_3, d_4) = (d_{1i}, d_{2i}, d_{3i}, d_{4i})$ satisfy $\min_{\overline{\Omega}} u_i \to 0$ as $i \to \infty$. Combining this with (3.3), we have $\max_{\overline{\Omega}} u_i \to 0$ and $u_i \to 0$ uniformly as $i \to \infty$.

We may assume, by passing to a subsequence if necessary, that $(d_{1i}, d_{2i}, d_{3i}, d_{4i}) \to (d_1, d_2, d_3, d_4)$ and $(u_i, v_i) \to (u, v)$ as $i \to \infty$, where both u and v are non-negative functions.

Let $\psi_i = v_i(1 + d_{4i} u_i)$, then the second equation of problem (1.3) can be written as

$$\begin{cases} -\Delta\psi_i = \dfrac{b - v_i + m_2 u_i}{d_{2i}(1 + d_{4i} u_i)}\psi_i, & x \in \Omega, \\ \dfrac{\partial\psi_i}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.5}$$

First, since $\|u_i\|_\infty \leqslant C_1$ and $\|v_i\|_\infty \leqslant C_2$, there is a positive constant C_3 independent of i such that $\|\psi_i\|_\infty \leqslant C_3$ and $\|\frac{b-v_i+m_2u_i}{d_{2i}(1+d_{4i}u_i)}\|_\infty \leqslant C_3$ for all $i \geqslant 1$. Thus, by making use of L^p-estimate to problem (3.5), we have $\|\psi_i\|_{W^{2,p}(\Omega)} \leqslant C_3^2|\Omega|^{\frac{1}{p}}$ for all $i \geqslant 1$, where $p > 1$ is any positive number. Next, let $p > N$, then, by the Sobolev embedding theorems, we have $\|\psi_i\|_{C^{1,\alpha}(\Omega)} \leqslant C^*\|\psi_i\|_{W^{2,p}(\Omega)} \leqslant C^*C_3^2|\Omega|^{\frac{1}{p}}$, where $\alpha \in (0,1)$ and C^* is the embedding constant. This, combined with u_i and $v_i \in C^2(\Omega)$, yields that $\frac{b-v_i+m_2u_i}{d_{2i}(1+d_{4i}u_i)}\psi_i \in C^\alpha(\Omega)$. And last, by the regularity theory for elliptic equations, we have $\psi_i \in C^{2,\alpha}(\Omega)$. Therefore, we can assume, by passing to a subsequence if necessary, that ψ_i converges uniformly to some function ψ in $C^2(\Omega)$. Combining this with $u_i \to 0$, we have $v_i = \frac{\psi_i}{1+d_{4i}u_i} \to \psi$ (or denoted by v) as $i \to \infty$. Furthermore, if $d_{2i} \to d_2 \in [D_2, \infty)$ as $i \to \infty$, then v satisfies

$$\begin{cases} -d_2\Delta v = v(b-v), & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Applying Lemma 3.1 to problem (3.6) and noting that $\min_{\overline{\Omega}} v_i \geqslant \frac{b}{C_4^*}$, we have $v = b$. If $d_{2i} \to \infty$ as $i \to \infty$, then ψ satisfies

$$\begin{cases} -\Delta\psi = 0, & x \in \Omega, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.7}$$

which implies that $\psi = v = C$ for some non-negative constant C.

Since $\int_\Omega v_i(b - v_i - m_2u_i)\,dx = 0$, by letting $i \to \infty$ and noting that $\min_{\overline{\Omega}} v_i \geqslant \frac{b}{C_4^*}$, $u_i \to 0$ and $v_i \to C$, we also have $v = b$.

Let the first equation of problem (1.3) be divided by $\max_{\overline{\Omega}} u_i$, then, by a similar argument as that in (3.5), we have $\frac{u_i}{\max_{\overline{\Omega}} u_i} \to \bar{u}$ uniformly in $C^2(\Omega)$. This, combined with $\frac{u_i}{\max_{\overline{\Omega}} u_i} \geqslant \frac{\min_{\overline{\Omega}} u_i}{\max_{\overline{\Omega}} u_i} \geqslant \frac{1}{C_3^*}$, yields that $\bar{u} \geqslant \frac{1}{C_3^*}$. On the other hand, the first equation of problem (1.3) is divided by $\max_{\overline{\Omega}} u_i$ and then integrated in Ω by parts

$$\int_\Omega \frac{u_i}{\max_{\overline{\Omega}} u_i}(a - u_i - m_1v_i)\,dx = 0. \tag{3.8}$$

Let $i \to \infty$, and note that $\frac{u_i}{\max_{\overline{\Omega}} u_i} \to \bar{u} \geqslant \frac{1}{C_3^*}$, $u_i \to 0$ and $v_i \to b$, then $a = m_1b$, which is a contradiction to $a \neq m_1b$. Therefore, $\min_{\overline{\Omega}} u(x) \geqslant C^{-1}$ if $a \neq m_1b$.

In conclusion, if necessary, we may suitably enlarge the upper positive bounds of both u and v, and diminish its lower positive bounds such that Lemma 3.3 holds. □

Lemma 3.4. *Let $a > m_1b$, and $(\tilde{u}, \tilde{v})^T$ be the constant positive solution of problem* (1.3). *If a satisfies*: (1) *$m_1b < a < \frac{2m_1b}{1-m_1m_2}$ when $0 < m_1m_2 < 1$, or* (2) *$a > m_1b$ when $m_1m_2 \geqslant 1$, then $\tilde{u} < m_1\tilde{v}$.*

Proof. The constant positive solution $(\tilde{u}, \tilde{v})^T$ of problem (1.3) satisfies

$$\begin{cases} a - \tilde{u} - m_1\tilde{v} = 0, \\ b - \tilde{v} + m_2\tilde{u} = 0. \end{cases} \tag{3.9}$$

It is obvious that, when $a > m_1 b$, (3.9) has the only positive solution as follows

$$\tilde{u} = \frac{a - m_1 b}{1 + m_1 m_2}, \qquad \tilde{v} = \frac{b + m_2 a}{1 + m_1 m_2}, \tag{3.10}$$

and $\tilde{u} - m_1 \tilde{v} = \frac{(1-m_1 m_2)a - 2m_1 b}{1+m_1 m_2}$. Therefore, if a satisfies: (1) $m_1 b < a < \frac{2m_1 b}{1-m_1 m_2}$ when $0 < m_1 m_2 < 1$, or (2) $a > m_1 b$ when $m_1 m_2 \geqslant 1$, then $\tilde{u} < m_1 \tilde{v}$. $\square$

Remark 3.2. For $0 < m_1 m_2 < 1$, if a satisfies $a \geqslant \frac{2m_1 b}{1-m_1 m_2}$, then we can check that problem (1.3) has no non-constant positive solution bifurcating from $(\tilde{u}, \tilde{v})^T$ (see Lemma 5.3(3)). Therefore, in the following sections, if a is not mentioned, then we always assume that a satisfies: (1) $m_1 b < a < \frac{2m_1 b}{1-m_1 m_2}$ when $0 < m_1 m_2 < 1$, or (2) $a > m_1 b$ when $m_1 m_2 \geqslant 1$.

4. Local analysis at the constant positive steady state

In this section, we study the linearization of problem (1.3) at $(\tilde{u}, \tilde{v})^T$. Its argument is similar to that of [13].

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition, $\mathbf{E}(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$, ϕ_{ij}, $j = 1, 2, 3, \ldots, \dim \mathbf{E}(\mu_i)$, be an orthonormal basis of $\mathbf{E}(\mu_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbf{R}^2\}$. Define

$$\begin{aligned}
&\mathbf{X} = \left\{ (u, v)^T \in \left[C^1(\overline{\Omega})\right]^2 \;\middle|\; \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0,\ x \in \partial\Omega \right\}, \\
&\mathbf{X}^+ = \left\{ (u, v)^T \in \mathbf{X} \;\middle|\; u > 0,\ v > 0,\ x \in \overline{\Omega} \right\}, \\
&\mathbf{B}(C) = \left\{ (u, v)^T \in \mathbf{X} \;\middle|\; C^{-1} < u, v < C,\ x \in \overline{\Omega} \right\},
\end{aligned}$$

then $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i$, where $\mathbf{X}_i = \bigoplus_{j=1}^{\dim \mathbf{E}(\mu_i)} \mathbf{X}_{ij}$. Let

$$\begin{aligned}
&\mathbf{\Phi}(u, v) = \big(d_1 u(1 + d_3 v), d_2 v(1 + d_4 u)\big)^T, \\
&\mathbf{G}(u, v) = \big(au - u^2 - m_1 uv, bv - v^2 + m_2 uv\big)^T,
\end{aligned}$$

then problem (1.3) can be written as

$$\begin{cases} -\Delta \mathbf{\Phi}(u, v) = \mathbf{G}(u, v), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Since

$$\frac{\partial \mathbf{\Phi}(u, v)}{\partial (u, v)} = \begin{pmatrix} d_1(1 + d_3 v) & d_1 d_3 u \\ d_2 d_4 v & d_2(1 + d_4 u) \end{pmatrix}$$

and $\det\{\frac{\partial \mathbf{\Phi}(u,v)}{\partial(u,v)}\} > 0$ for all non-negative $(u, v)^T$, $[\frac{\partial \mathbf{\Phi}(u,v)}{\partial(u,v)}]^{-1}$ exists and $\det\{[\frac{\partial \mathbf{\Phi}(u,v)}{\partial(u,v)}]^{-1}\}$ is positive. Hence $\mathbf{w} \triangleq (u, v)^T$ is a positive solution of problem (4.1) if and only if

$$\mathbf{F}(\mathbf{w}) \triangleq \mathbf{w} - (\mathbf{I} - \Delta)^{-1}\left\{\left[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w})\right]^{-1}\left[\mathbf{G}(\mathbf{w}) + \nabla \mathbf{w} \mathbf{\Phi}_{\mathbf{ww}}(\mathbf{w}) \nabla \mathbf{w}\right] + \mathbf{w}\right\} = 0, \quad \mathbf{w} \in \mathbf{X}^+, \tag{4.2}$$

where $(\mathbf{I}-\Delta)^{-1}$ is the inverse of $\mathbf{I}-\Delta$ in $\mathbf{X}$. As $\mathbf{F}(\cdot)$ is a compact perturbation of the identity operator, for any $\mathbf{B} \triangleq \mathbf{B}(C)$, the Leray–Schauder degree $\deg(\mathbf{F}(\cdot), \mathbf{0}, \mathbf{B})$ is well-defined if $\mathbf{F}(\mathbf{w}) \neq 0$ on $\partial\mathbf{B}$. Further, we note that

$$D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}}) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\big\{\big[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\big]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{I}\big\}, \tag{4.3}$$

where $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{v})^T$. We recall that if $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$ is invertible, the index of $\mathbf{F}$ at $\tilde{\mathbf{w}}$ is defined as $\operatorname{index}(\mathbf{F}(\cdot), \tilde{\mathbf{w}}) = (-1)^r$, where r is the number of negative eigenvalues of $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$.

We refer to the decomposition in our discussion of the eigenvalues of $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$. First, we note that, for each integer $i \geqslant 1$ and each integer $1 \leqslant j \leqslant \dim E(\mu_i)$, $\mathbf{X}_{ij}$ is invariant under $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$, namely, $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})\mathbf{w} \in \mathbf{X}_{ij}$ for all $\mathbf{w} \in \mathbf{X}_{ij}$. Thus, λ is an eigenvalue of $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$ on $\mathbf{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$\mathbf{I} - \frac{1}{1+\mu_i}\big\{\big[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\big]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{I}\big\} = \frac{1}{1+\mu_i}\big\{\mu_i\mathbf{I} - \big[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\big]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\big\}. \tag{4.4}$$

Since $\det\{\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\} \neq 0$ when $a > m_1 b$, $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$ is invertible if and only if, for all $i \geqslant 1$, the matrix

$$\mathbf{I} - \frac{1}{1+\mu_i}\big\{\big[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\big]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{I}\big\} \tag{4.5}$$

is non-singular. Writing

$$H(\mu) \triangleq H(\tilde{\mathbf{w}}, \mu) = \det\big[\mu I - \big[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\big]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\big], \tag{4.6}$$

we note, furthermore, that the sign of $\det\{\mathbf{I} - \frac{1}{1+\mu_i}\{[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) + I\}\}$ depends on the number of negative eigenvalue of $\mathbf{I} - \frac{1}{1+\mu_i}\{[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{I}\}$, and both $H(\mu_i)$ and $\det\{\mathbf{I} - \frac{1}{1+\mu_i}\{[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) + \mathbf{I}\}\}$ have the same signs. Therefore, if $H(\mu_i) \neq 0$, then for each integer $1 \leqslant j \leqslant \dim E(\mu_i)$, the number of negative eigenvalues of $D_{\mathbf{w}}\mathbf{F}(\tilde{\mathbf{w}})$ on $\mathbf{X}_{ij}$ is odd if and only if $H(\mu_i) < 0$.

In conclusion, we have the following:

Lemma 4.1. *Suppose that, for all $i \geqslant 1$, the matrix $\mu_i\mathbf{I} - [\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})]^{-1}\mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})$ is non-singular. Then*

$$\operatorname{index}\big(\mathbf{F}(\cdot), \tilde{\mathbf{w}}\big) = (-1)^r, \quad \textit{where } r = \sum_{i \geqslant 1,\, H(\mu_i) < 0} \dim \mathbf{E}(\mu_i).$$

To facilitate our computation of $\operatorname{index}(F(\cdot), \tilde{\mathbf{w}})$, we will consider carefully the sign of $H(\mu_i)$. Note that

$$H(\mu) = \det\big\{\big[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})\big]^{-1}\big\}\det\big\{\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) - \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\big\}, \tag{4.7}$$

and $\det\{[\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}})]^{-1}\} > 0$, so we will only need to consider $\det\{\mu_i\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) - \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\}$.

In the following, by analyzing the two roots of $H(\mu) = 0$, we discuss the sign of $H(\mu_i)$. As

$$\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) = \begin{pmatrix} d_1(1+d_3\tilde{v}) & d_1 d_3 \tilde{u} \\ d_2 d_4 \tilde{v} & d_2(1+d_4\tilde{u}) \end{pmatrix}, \qquad \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}}) = \begin{pmatrix} -\tilde{u} & -m_1\tilde{u} \\ m_2\tilde{v} & -\tilde{v} \end{pmatrix},$$

we have

$$\det\big\{\mu\mathbf{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) - \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\big\} = A\mu^2 + B\mu + C, \tag{4.8}$$

where

$$A = d_1 d_2(1 + d_3\tilde{v} + d_4\tilde{u}) > 0,$$
$$B = d_1(1 + d_3\tilde{v})\tilde{v} + d_2\tilde{u}\big[1 + d_4(\tilde{u} - m_1\tilde{v})\big] + d_1 d_3 m_2 \tilde{u}\tilde{v},$$
$$C = \tilde{u}\tilde{v} + m_1 m_2 \tilde{u}\tilde{v} > 0.$$

When $B^2 - 4AC > 0$, let $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ denote two roots of $\det\{\mu\boldsymbol{\Phi}_{\mathbf{w}}(\tilde{\mathbf{w}}) - \mathbf{G}_{\mathbf{w}}(\tilde{\mathbf{w}})\} = 0$ with $\mu_1(\mathbf{d}) < \mu_2(\mathbf{d})$. We note that $m_1\tilde{v} > \tilde{u}$ when a satisfies the assumption in Remark 3.2. Therefore, if $d_4 \leqslant \frac{1}{m_1\tilde{v}-\tilde{u}}$, then $B > 0$, and both $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ are non-positive. Whereas if both $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ are positive and $\mu_1(\mathbf{d}) \neq \mu_2(\mathbf{d})$, then it is necessary that $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$ and $B^2 - 4AC > 0$.

Now, we discuss the dependence of $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ on d_i, $i = 1, 2, 3, 4$.

Lemma 4.2. *Assume that $d_1 > 0, d_2 > 0, d_3 \geqslant 0$ and $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$ satisfying $B^2 - 4AC > 0$.*

(i) *Let d_2, d_3 and d_4 be fixed, then*

$$\lim_{d_1\to 0^+} \mu_1(\mathbf{d}) = \frac{\tilde{u}\tilde{v} + m_1 m_2 \tilde{u}\tilde{v}}{d_2\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]}, \qquad \lim_{d_1\to 0^+} \mu_2(\mathbf{d}) = +\infty.$$

Furthermore, $0 < \lim_{d_1\to 0^+} \mu_1(\mathbf{d}) < \mu_1$ when one of d_2 and d_4 is large enough.

(ii) *Let d_1, d_3 and d_4 be fixed, then there exists a positive constant $D_2(d_1, d_3, d_4)$ such that $0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) < C(d_1)$ for all $d_2 > D_2(d_1, d_3, d_4)$.*

(iii) *Let d_1, d_2 and d_3 be fixed, then there exists a positive constant $D_4(d_1, d_2, d_3)$ such that $0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) < C(d_1)$ for all $d_4 > D_4(d_1, d_2, d_3)$.*

Proof. (i) We note that $B < 0$ when $d_1 < \frac{d_2\tilde{u}[d_4(m_1\tilde{v}-\tilde{u})-1]}{(1+d_3\tilde{v})\tilde{v}+d_3 m_2\tilde{u}\tilde{v}}$, thus both $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ are positive. Let $A_1 = d_2(1 + d_3\tilde{v} + d_4\tilde{u})$, $B_1 = (1 + d_3\tilde{v})\tilde{v} + d_3 m_2\tilde{u}\tilde{v}$, $B_2 = d_2\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]$ and $B = B_1 d_1 - B_2$, then

$$\mu_1(\mathbf{d}) = \frac{2C}{B_2 - B_1 d_1 + \sqrt{(B_2 - B_1 d_1)^2 - 4 d_1 A_1 C}}, \tag{4.9}$$

$$\mu_2(\mathbf{d}) = \frac{B_2 - B_1 d_1 + \sqrt{(B_2 - B_1 d_1)^2 - 4 d_1 A_1 C}}{2 d_1 A_1}. \tag{4.10}$$

Let d_2, d_3 and d_4 be fixed. Since $B_2 - d_1 B_1 > 0$, we have

$$\lim_{d_1\to 0^+} \mu_1(\mathbf{d}) = \frac{\tilde{u}\tilde{v} + m_1 m_2 \tilde{u}\tilde{v}}{d_2\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]},$$
$$\lim_{d_1\to 0^+} \mu_2(\mathbf{d}) \geqslant \lim_{d_1\to 0^+} \frac{B_2}{2 d_1 A_1} - \frac{B_1}{2A_1} \to +\infty. \tag{4.11}$$

Therefore, if one of d_2 and d_4 is large enough, then $0 < \lim_{d_1\to 0^+} \mu_1(\mathbf{d}) < \mu_1$.

(ii) We note that $B < 0$ when $d_2 > \frac{d_1(1+d_3\tilde{v})\tilde{v}+d_1 d_3 m_2\tilde{u}\tilde{v}}{\tilde{u}[d_4(m_1\tilde{v}-\tilde{u})-1]}$, thus both $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ are positive. Let d_1, d_3 and d_4 be fixed. Since $\lim_{d_2\to\infty} \frac{A}{d_2} = d_1(1 + d_3\tilde{v} + d_4\tilde{u})$, $\lim_{d_2\to\infty} \frac{B}{d_2} = \tilde{u}[1 + d_4(\tilde{u} - m_1\tilde{v})]$ and $\lim_{d_2\to\infty} \frac{C}{d_2} = 0$, we have

$$\lim_{d_2\to\infty} \mu_1(\mathbf{d}) = 0, \qquad \lim_{d_2\to\infty} \mu_2(\mathbf{d}) = \frac{\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]}{d_1(1 + d_3\tilde{v} + d_4\tilde{u})}. \tag{4.12}$$

Furthermore, by making use of (4.10), we have

$$\mu_2(\mathbf{d}) < \frac{B_2}{d_1 A_1} = \frac{\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]}{d_1(1 + d_3\tilde{v} + d_4\tilde{u})} < \frac{m_1\tilde{v} - \tilde{u}}{d_1} \triangleq C(d_1).$$

Therefore, there exists a positive constant $D_2(d_1, d_3, d_4)$ such that $0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) < C(d_1)$ for all $d_2 > D_2(d_1, d_3, d_4)$.

(iii) We note that $B < 0$ when $d_4 > \frac{d_1(1+d_3\tilde{v})\tilde{v}+d_2\tilde{u}+d_1d_3m_2\tilde{u}\tilde{v}}{d_2\tilde{u}(m_1\tilde{v}-\tilde{u})}$, thus both $\mu_1(\mathbf{d})$ and $\mu_2(\mathbf{d})$ are positive. Let d_1, d_2 and d_3 be fixed. Since $\lim_{d_4\to\infty} \frac{A}{d_4} = d_1 d_2\tilde{u}$, $\lim_{d_4\to\infty} \frac{B}{d_4} = d_2\tilde{u}(\tilde{u} - m_1\tilde{v})$ and $\lim_{d_4\to\infty} \frac{C}{d_4} = 0$, we have

$$\lim_{d_4\to\infty} \mu_1(\mathbf{d}) = 0, \qquad \lim_{d_4\to\infty} \mu_2(\mathbf{d}) = \frac{m_1\tilde{v} - \tilde{u}}{d_1}, \tag{4.13}$$

which implies that there exists a positive constant $D_4(d_1, d_2, d_3)$ satisfying $D_4(d_1, d_2, d_3) \geqslant \frac{1}{m_1\tilde{v}-\tilde{u}}$ such that $0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) < C(d_1)$ for all $d_4 > D_4(d_1, d_2, d_3)$.

Last, let d_1, d_2 and d_4 be fixed. If $B < 0$, then $d_3 < \frac{d_2\tilde{u}[d_4(m_1\tilde{v}-\tilde{u})-1]-d_1\tilde{v}}{d_1\tilde{v}^2+d_1m_2\tilde{u}\tilde{v}}$. □

To sum up, we have

Lemma 4.3. *If* $0 < \mu_1(\mathbf{d}) < \mu_i < \mu_2(\mathbf{d})$ *for some* $i \geqslant 1$, *then* $H(\mu_i) < 0$. *Whereas,* $H(\mu_i) > 0$ *provided* $\mu_i \notin [\mu_1(\mathbf{d}), \mu_2(\mathbf{d})]$ *for some* $i \geqslant 0$.

Remark 4.1.

(i) Let d_2, d_3, d_4 be fixed and satisfy that $d_2 > 0$, $d_3 \geqslant 0$, $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$, one of d_2 and d_4 be large enough. If d_1 is also large enough, then $0 < \mu_1(\mathbf{d}) \leqslant \mu_2(\mathbf{d}) < \mu_1$. In this case, we may prove that problem (1.3) has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$, (see Lemma 5.3(2)).

(ii) For any $n \geqslant 1$ given, we may choose d_1, d_2, d_3 and d_4 satisfying that d_1 is a positive number small enough, $d_2 > 0$, $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$, and one of d_2 and d_4 is large enough, such that $0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) \in (\mu_n, \mu_{n+1})$.

5. Non-existence of non-constant positive solution

In this section, we discuss the non-existence of non-constant positive solution of problem (1.3). We mainly use the methods in [10,13]. To this end, we first introduce the definitions of bifurcation and regular points and the bifurcation theory. See also [4,12,13].

Definition 5.1. Suppose that $a > m_1 b$ and $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{v})^T$. Let d_2, d_3 and d_4 be fixed, and $\tilde{d}_1 \in (0, +\infty)$. $(\tilde{d}_1, \tilde{\mathbf{w}})$ is a bifurcation point of problem (1.3) if, for any $\delta \in (0, \tilde{d}_1)$, there exists $d_1 \in [\tilde{d}_1 - \delta, \tilde{d}_1 + \delta]$ such that problem (1.3) has a non-constant positive solution in $\mathbf{B}_\delta(\tilde{\mathbf{w}})$. Otherwise, we say that $(\tilde{d}_1, \tilde{\mathbf{w}})$ is a regular point. See [17].

Bifurcation and regular points with respect to d_2, d_3 and d_4, respectively, are defined analogously.

Lemma 5.1. *Let* $\Gamma_p = \{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\}$, $\mathbf{S}_{d_1}(d_1) = \{\mu \geqslant 0 \mid H(\mu) = 0\}$, *where* $H(\mu)$ *is as defined in* (4.6), *and let* A *and* C *be as defined in* (4.8). *We note that, for every* d_1, *since* $A > 0$ *and* $C > 0$, $\mathbf{S}_{d_1}(d_1)$ *has* 0 *or* 2 *elements.*

The bifurcation theory [4] *indicates that*:

(1) *if* $\Gamma_p \cap \mathbf{S}_{d_1}(\tilde{d}_1) = \emptyset$, *then* $(\tilde{d}_1, \tilde{\mathbf{w}})$ *is a regular point of problem* (1.3);
(2) *if* $\Gamma_p \cap \mathbf{S}_{d_1}(\tilde{d}_1) \neq \emptyset$, *then* $(\tilde{d}_1, \tilde{\mathbf{w}})$ *may be a bifurcation point of problem* (1.3).

Now, we discuss the non-existence of non-constant positive solution to problem (1.3), and note that $\tilde{u} < m_1\tilde{v}$ if a satisfies the assumption of Remark 3.2.

Lemma 5.2. *Suppose that a satisfies the assumption of Remark* 3.2.

(1) *Let* D_1, D_2 *and* D_3 *be positive constants given, where* D_1 *and* D_2 *are small enough, and* D_3 *is large enough, then there exist positive constants* D_2^0 *and* D_4^0 *satisfying* $D_2^0 \geqslant D_2$ *and* $D_4^0 \leqslant \frac{1}{m_1\tilde{v}-\tilde{u}}$ *such that, when* $d_1 \geqslant D_1$, $d_2 > D_2^0$, $d_3 \leqslant D_3$ *and* $d_4 < D_4^0$, *problem* (1.3) *has no non-constant positive solution.*
(2) *If* $d_4 \leqslant \frac{1}{m_1\tilde{v}-\tilde{u}}$, *then* (1.3) *has no non-constant positive solution bifurcating from* $\tilde{\mathbf{w}}$.

Proof. (1) Its proof is similar to that of Theorem 3.1 in [10]. For the completeness of our paper, we still give a complete procedure to its proof.

Assume, on the contrary, that there exists a sequence of $\{(d_{1i}, d_{2i}, d_{3i}, d_{4i})\}$ satisfying $d_{1i} \geqslant D_1$, $d_{2i} \geqslant D_2$, $d_{3i} \leqslant D_3$, $d_{4i} \leqslant \frac{1}{m_1\tilde{v}-\tilde{u}}$, $d_{2i} \to \infty$, $d_{4i} \to 0$, such that $(u_i, v_i)^T$ are non-constant positive solutions to problem (1.3) with $(d_1, d_2, d_3, d_4) = (d_{1i}, d_{2i}, d_{3i}, d_{4i})$.

Set $\phi_i = u_i(1 + d_{3i}v_i)$, $\psi_i = v_i(1 + d_{4i}u_i)$. The second equation of (1.3) is written as

$$\begin{cases} -\Delta\psi_i = \dfrac{b - v_i + m_2u_i}{d_{2i}(1 + d_{4i}u_i)}\psi_i, & x \in \Omega, \\ \dfrac{\partial\psi_i}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

By the same argument as (3.5), we may assume, by passing to a subsequence if necessary, that ψ_i converges uniformly to some function ψ in $C^2(\Omega)$. Furthermore, ψ satisfies

$$\begin{cases} -\Delta\psi = 0, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases} \tag{5.2}$$

which implies that $\psi = \psi_0$ for some non-negative constant ψ_0. Hence,

$$|v_i - \psi_0| \leqslant |\psi_i - \psi_0| + d_{4i}\|u_i\|_\infty\|v_i\|_\infty \to 0.$$

Set $\bar{v}_i = \frac{v_i}{\|v_i\|_\infty}$. By $\min_{\overline{\Omega}} v_i \geqslant \frac{b}{C_4^*}$ and $v_i \to \psi_0$, we obtain that $\bar{v}_i \to 1$ uniformly.

On the other hand, ϕ_i satisfies the following equation

$$\begin{cases} -\Delta\phi_i = \dfrac{\phi_i}{d_{1i}(1 + d_{3i}v_i)}\left(a - \dfrac{\phi_i}{1 + d_{3i}v_i} - m_1v_i\right), & x \in \Omega, \\ \dfrac{\partial\phi_i}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

By a similar argument, there exists a subsequence which still is denoted by $\{\phi_i\}$, and ϕ such that $\phi_i \to \phi$ uniformly in $C^2(\Omega)$. Since $d_{3i}\|v_i\|_\infty \leqslant D_3C_2$, we may assume, by passing to a subsequence if necessary, that $d_{3i}\|v_i\|_\infty \to \tau$. Applying Lemma 2.1 to (5.3), we have

$$\Bigl(1+d_{3i}\|v_i\|_\infty \min_{\overline{\Omega}} \bar{v}_i\Bigr)\Bigl(a-m_1 \max_{\overline{\Omega}} v_i\Bigr)$$
$$\leqslant \phi_i \leqslant \Bigl(1+d_{3i}\|v_i\|_\infty \max_{\overline{\Omega}} \bar{v}_i\Bigr)\Bigl(a-m_1 \min_{\overline{\Omega}} v_i\Bigr). \tag{5.4}$$

Since $d_{3i}\|v_i\|_\infty \to \tau$, $v_i \to \psi_0$ and $\bar{v}_i \to 1$, we see that $\phi_i \to (1+\tau)(a-m_1\psi_0)$ uniformly. Therefore, by $d_{3i} \to d_3$ and $v_i \to \psi_0$, we have $u_i \to u$ for some non-negative constant u.

If $u \equiv 0$, then, by integrating the second equation of problem (1.3) in Ω, letting $i \to \infty$ and noting that $\min_{\overline{\Omega}} v_i \geqslant \frac{b}{C_4^*}$, we have $v_i \to b$. By a similar argument as that of (3.8), we have $a = m_1 b$, which is a contradiction to $a > m_1 b$. Hence $u = \tilde{u}$ and $v = \tilde{v}$.

We further show that $(u_i, v_i)^T = (\tilde{u}, \tilde{v})^T$ for all large i. By the definition of $(\phi_i, \psi_i)^T$, we know that both u_i and v_i are functions of $(\phi_i, \psi_i)^T$, and $(\phi_i, \psi_i)^T$ satisfies

$$\begin{cases} -d_{1i}\Delta\phi_i = u_i(\phi_i,\psi_i)\bigl(a-u_i(\phi_i,\psi_i)-m_1 v_i(\phi_i,\psi_i)\bigr), & x \in \Omega, \\ -d_{2i}\Delta\psi_i = v_i(\phi_i,\psi_i)\bigl(b-v_i(\phi_i,\psi_i)+m_2 u_i(\phi_i,\psi_i)\bigr), & x \in \Omega, \\ \dfrac{\partial \phi_i}{\partial \nu} = \dfrac{\partial \psi_i}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{5.5}$$

It is easy to check that

$$\frac{\partial u}{\partial \phi} = \frac{1+d_4 u}{1+d_3 v+d_4 u}, \qquad \frac{\partial v}{\partial \phi} = -\frac{d_4 v}{1+d_3 v+d_4 u},$$
$$\frac{\partial u}{\partial \psi} = -\frac{d_3 u}{1+d_3 v+d_4 u}, \qquad \frac{\partial v}{\partial \psi} = \frac{1+d_3 v}{1+d_3 v+d_4 u}.$$

Let $\bar{\phi}_i = \frac{1}{|\Omega|}\int_\Omega \phi_i\,dx$ and $\overline{\psi}_i = \frac{1}{|\Omega|}\int_\Omega \psi_i\,dx$ denote the average of ϕ_i and ψ_i, respectively. Set

$$h(\phi_i,\psi_i) = u_i(\phi_i,\psi_i)\bigl(a-u_i(\phi_i,\psi_i)-m_1 v_i(\phi_i,\psi_i)\bigr).$$

Multiplying the first equation of (5.5) by $\phi_i - \bar{\phi}_i$ and integrating in Ω, we have

$$\begin{aligned} d_{1i}&\int_\Omega |\nabla\phi_i|^2\,dx \\ &= \int_\Omega h(\phi_i,\psi_i)(\phi_i-\bar{\phi}_i)\,dx \\ &= \int_\Omega \bigl\{\bigl[h(\phi_i,\psi_i)-h(\bar{\phi}_i,\psi_i)\bigr]+\bigl[h(\bar{\phi}_i,\psi_i)-h(\bar{\phi}_i,\overline{\psi}_i)\bigr]\bigr\}(\phi_i-\bar{\phi}_i)\,dx \\ &\leqslant \int_\Omega \frac{\partial h}{\partial \phi}(\xi_i,\psi_i)(\phi_i-\bar{\phi}_i)^2\,dx + \int_\Omega \left|\frac{\partial h}{\partial \psi}(\bar{\phi}_i,\eta_i)\right| |\phi_i-\bar{\phi}_i||\psi_i-\overline{\psi}_i|\,dx, \end{aligned} \tag{5.6}$$

where ξ_i lies between ϕ_i and $\bar{\phi}_i$, and η_i lies between ψ_i and $\overline{\psi}_i$. Since $(u_i, v_i)^T \to (\tilde{u}, \tilde{v})^T$, and both $\phi_i - \bar{\phi}_i$ and $\psi_i - \overline{\psi}_i$ converge to zero uniformly, we can check that

$$\begin{aligned} &\frac{\partial h}{\partial \phi}(\xi_i,\psi_i) + \frac{\tilde{u}}{1+d_{3i}\tilde{v}} \\ &\quad = \frac{(a-2u_i-m_1 v_i)(1+d_{4i}u_i)}{1+d_{3i}v_i+d_{4i}u_i} + \frac{m_1 d_{4i} u_i v_i}{1+d_{3i}v_i+d_{4i}u_i} + \frac{\tilde{u}}{1+d_{3i}\tilde{v}} + o(1) \to 0, \end{aligned} \tag{5.7}$$

$$\begin{aligned}\frac{\partial h}{\partial \psi}(\overline{\phi}_i,\eta_i)&=-\frac{(a-2u_i-m_1v_i)d_{3i}u_i}{1+d_{3i}v_i+d_{4i}u_i}-\frac{m_1u_i(1+d_{3i}v_i)}{1+d_{3i}v_i+d_{4i}u_i}+o(1)\\&\to\frac{d_3\tilde u^2-m_1\tilde u(1+d_3\tilde v)}{1+d_3\tilde v},\end{aligned}\tag{5.8}$$

uniformly. Therefore, there exists $i_0>0$ such that $\frac{\partial h}{\partial \phi}(\xi_i,\psi_i)<-\frac{\tilde u}{2(1+d_{3i}\tilde v)}$ for all $i>i_0$.

On other hand, since $d_{3i}\leqslant D_3$, $\|u_i\|_\infty\leqslant C_1$ and $\|v_i\|_\infty\leqslant C_2$, we have $\|\frac{\partial h}{\partial \psi}(\overline{\phi}_i,\eta_i)\|_\infty\leqslant C$ for all i, where C is a positive constant independent of i. Substituting these results into (5.6) and using the Young's inequality, we have

$$d_{1i}\int_\Omega|\nabla\phi_i|^2\,dx\leqslant\left(-\frac{\tilde u}{2(1+d_{3i}\tilde v)}+\varepsilon\right)\int_\Omega(\phi_i-\overline{\phi}_i)^2\,dx+\frac{C^2}{\varepsilon}\int_\Omega(\psi_i-\overline{\psi}_i)^2\,dx.\tag{5.9}$$

Set $g(\phi_i,\psi_i)=v_i(\phi_i,\psi_i)(b-v_i(\phi_i,\psi_i)+m_2u_i(\phi_i,\psi_i))$. By a similar argument, we have

$$\frac{\partial g}{\partial \phi}(\xi_i,\overline{\psi}_i)=\frac{m_2v_i(1+d_{4i}u_i)}{1+d_{3i}v_i+d_{4i}u_i}-\frac{(b-2v_i+m_2u_i)d_{4i}v_i}{1+d_{3i}v_i+d_{4i}u_i}+o(1)\to\frac{m_2\tilde v}{1+d_3\tilde v},\tag{5.10}$$

$$\begin{aligned}\frac{\partial g}{\partial \psi}(\phi_i,\eta_i)&=-\frac{m_2v_id_{3i}u_i}{1+d_{3i}v_i+d_{4i}u_i}+\frac{(b-2v_i+m_2u_i)(1+d_{3i}v_i)}{1+d_{3i}v_i+d_{4i}u_i}+o(1)\\&\to-\frac{m_2d_3\tilde u\tilde v}{1+d_3\tilde v}-\tilde v.\end{aligned}\tag{5.11}$$

Therefore, $\|\frac{\partial g}{\partial \psi}(\xi_i,\overline{\psi}_i)\|_\infty\leqslant C$, $\|\frac{\partial g}{\partial \psi}(\phi_i,\eta_i)\|_\infty\leqslant C$, where C is a positive constant independent of i. Multiplying the second equation of problem (5.5) by $\psi_i-\overline{\psi}_i$ and integrating in Ω, by similar arguments as (5.6) and (5.9), we have

$$d_{2i}\int_\Omega|\nabla\psi_i|^2\,dx\leqslant\varepsilon\int_\Omega(\phi_i-\overline{\phi}_i)^2\,dx+\left(\frac{C^2}{\varepsilon}+C\right)\int_\Omega(\psi_i-\overline{\psi}_i)^2\,dx.\tag{5.12}$$

Let $\varepsilon=\frac{\tilde u}{4(1+d_{3i}\tilde v)}$. It follows from (5.9) and (5.12) that

$$d_{1i}\int_\Omega|\nabla\phi_i|^2\,dx+d_{2i}\int_\Omega|\nabla\psi_i|^2\,dx\leqslant\left(\frac{8(1+D_3\tilde v)C^2}{\tilde u}+C\right)C^{*2}\int_\Omega|\nabla\psi_i|^2\,dx,\tag{5.13}$$

where C^* is the Poincaré's embedding constant, i.e., C^* satisfies that $\|\psi_i-\overline{\psi}_i\|_{L^2(\Omega)}\leqslant C^*\|\psi_i\|_{W^{1,2}(\Omega)}$ for all $\psi_i\in W^{1,2}(\Omega)$. If necessary, we enlarge i_0 such that $d_{2i}>(\frac{8(1+D_3\tilde v)C^2}{\tilde u}+C)C^{*2}$ for all $i>i_0$, then $\nabla\phi_i=\nabla\psi_i=0$ for all $i>i_0$, which implies that ϕ_i and ψ_i are constants. Hence $u_i=\tilde u$ and $v_i=\tilde v$ for all $i>i_0$. We choose D_2^0 and the corresponding D_4^0 satisfying $D_2^0\geqslant(\frac{8(1+D_3\tilde v)C^2}{\tilde u}+C)C^{*2}$ and $D_4^0\leqslant\frac{1}{m_1\tilde v-\tilde u}$, then when $d_1\geqslant D_1$, $d_2>D_2^0$, $d_3\leqslant D_3$ and $d_4<D_4^0$, problem (1.3) has no non-constant positive solution.

(2) Let d_1 be extended to $(0,+\infty)$, d_2 to $(0,+\infty)$, d_3 to $[0,+\infty)$ and d_4 to $[0,\frac{1}{m_1\tilde v-\tilde u}]$, respectively. When $d_4\leqslant\frac{1}{m_1\tilde v-\tilde u}$, by (4.7) and (4.8), we have $H(\mu)>0$ for all $\mu\geqslant0$. Thus $\Gamma_p\cap\mathbf{S}_{d_j}(d_j)=\emptyset$, $j=1,2,3,4$. Therefore, by using Lemma 5.1, all these points $(d_j,\tilde{\mathbf{w}})$ are regular points of problem (1.3). This, combined with Lemma 5.2(1), yields that problem (1.3) has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$ if $d_4\leqslant\frac{1}{m_1\tilde v-\tilde u}$. □

Lemma 5.3.

(1) *Suppose that $a > m_1 b$. Let D_1, D_2, D_3 and D_4 be positive constants given, where D_1 and D_2 are small enough, and D_3 and D_4 are large enough, then there exist positive constants D_1^0 and D_3^0 satisfying $D_1^0 \geqslant D_1$ and $D_3^0 \leqslant D_3$ such that, when $d_1 > D_1^0$, $d_2 \geqslant D_2$, $d_3 < D_3^0$ and $d_4 \leqslant D_4$, problem* (1.3) *has no non-constant positive solution.*
(2) *Let a satisfy the assumption of Remark* 3.2. *If $d_1 \geqslant \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, then problem* (1.3) *has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$.*
(3) *For $0 < m_1 m_2 < 1$, if $a \geqslant \frac{2m_1 b}{1-m_1m_2}$, then problem* (1.3) *has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$.*

Proof. (1) Its proof is similar to that of Lemma 5.2(1). Here we give a simple proof.

Assume, on the contrary, that there exists a sequence of $\{(d_{1i}, d_{2i}, d_{3i}, d_{4i})\}$ satisfying $d_{1i} \geqslant D_1$, $d_{2i} \geqslant D_2$, $d_{3i} \leqslant D_3$, $d_{4i} \leqslant D_4$, $d_{1i} \to \infty$ and $d_{3i} \to 0$, such that $(u_i, v_i)^T$ are non-constant positive solutions of problem (1.3) with $(d_1, d_2, d_3, d_4) = (d_{1i}, d_{2i}, d_{3i}, d_{4i})$.

Set $\phi_i = u_i(1 + d_{3i}v_i)$, $\psi_i = v_i(1 + d_{4i}u_i)$. It is easy to check that $u_i \to \tilde{u}$ and $v_i \to \tilde{v}$ as $i \to \infty$. Thus, by making use of the equality of (5.11), there exists $i_0 > 0$ such that $\frac{\partial g}{\partial \psi}(\phi_i, \eta_i) < -\frac{\tilde{v}(1+d_{3i}\tilde{v})}{2(1+d_{3i}\tilde{v}+d_{4i}\tilde{u})}$ for all $i > i_0$. By similar arguments as (5.6), (5.9) and (5.12), we have

$$d_{1i}\int_\Omega |\nabla\phi_i|^2\,dx \leqslant \left(C + \frac{C^2}{\varepsilon}\right)\int_\Omega (\phi_i - \overline{\phi}_i)^2\,dx + \varepsilon\int_\Omega (\psi_i - \overline{\psi}_i)^2\,dx, \tag{5.14}$$

$$d_{2i}\int_\Omega |\nabla\psi_i|^2\,dx \leqslant \frac{C^2}{\varepsilon}\int_\Omega (\phi_i - \overline{\phi}_i)^2\,dx + \left(\varepsilon - \frac{\tilde{v}(1+d_{3i}\tilde{v})}{2(1+d_{3i}\tilde{v}+d_{4i}\tilde{u})}\right)\int_\Omega (\psi_i - \overline{\psi}_i)^2\,dx, \tag{5.15}$$

where C, which is independent of i, is an upper bound of $\|\frac{\partial h}{\partial \phi}(\xi_i, \psi_i)\|_\infty$, $\|\frac{\partial h}{\partial \psi}(\overline{\phi}_i, \eta_i)\|_\infty$ and $\|\frac{\partial g}{\partial \phi}(\xi_i, \overline{\psi}_i)\|_\infty$. Let $\varepsilon = \frac{\tilde{v}(1+d_{3i}\tilde{v})}{4(1+d_{3i}\tilde{v}+d_{4i}\tilde{u})}$. It follows from (5.14) and (5.15) that

$$d_{1i}\int_\Omega |\nabla\phi_i|^2\,dx + d_{2i}\int_\Omega |\nabla\psi_i|^2\,dx \leqslant \left(C + \frac{8(1+D_3\tilde{v}+D_4\tilde{u})C^2}{\tilde{v}}\right)C^{*2}\int_\Omega |\nabla\phi_i|^2\,dx. \tag{5.16}$$

If necessary, we enlarge i_0 such that $d_{1i} > (C + \frac{8(1+D_3\tilde{v}+D_4\tilde{u})C^2}{\tilde{v}})C^{*2}$ for all $i > i_0$, then $\nabla\phi_i = \nabla\psi_i = 0$ for all $i > i_0$, which implies that both ϕ_i and ψ_i are constants. Hence $u_i = \tilde{u}$ and $v_i = \tilde{v}$ for all $i > i_0$. We may choose positive constants D_1^0 and the corresponding D_3^0 satisfying $D_1^0 \geqslant (C + \frac{8(1+D_3\tilde{v}+D_4\tilde{u})C^2}{\tilde{v}})C^{*2}$ and $D_3^0 \leqslant D_3$, then when $d_1 > D_1^0$, $d_2 \geqslant D_2$, $d_3 < D_3^0$ and $d_4 \leqslant D_4$, problem (1.3) has no non-constant positive solution.

(2) Let d_1 be extended to $[\frac{m_1\tilde{v}-\tilde{u}}{\mu_1}, \infty)$, d_2 to $(0, +\infty)$, d_3 to $[0, +\infty)$ and d_4 to $[0, +\infty)$, respectively, and let A, B, C, A_1, B_1, B_2 and $H(\mu)$ be as defined in Section 4. If $B^2 - 4AC \geqslant 0$, then, by $A > 0$ and $C > 0$, and by making use of (4.9) and (4.10), we have

$$0 < \mu_1(\mathbf{d}) \leqslant \mu_2(\mathbf{d}) < \frac{B_2}{d_1A_1} = \frac{\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]}{d_1(1 + d_3\tilde{v} + d_4\tilde{u})} < \frac{m_1\tilde{v} - \tilde{u}}{d_1} \leqslant \mu_1, \tag{5.17}$$

or $\mu_1(\mathbf{d}) \leqslant \mu_2(\mathbf{d}) < 0$, which implies that $\mu_i \notin [\mu_1(\mathbf{d}), \mu_2(\mathbf{d})]$ for all $i \geqslant 0$. Thus, by Lemma 4.3, we have $H(\mu_i) > 0$ for all $i \geqslant 0$.

If $B^2 - 4AC < 0$, then, by $A > 0$ and $C > 0$, we also have $H(\mu_i) > 0$ for all $i \geqslant 0$.

In conclusion, we have $\Gamma_p \cap \mathbf{S}_{d_j}(d_j) = \emptyset$, $j = 1, 2, 3, 4$. Thus, by Lemma 5.1, all of $(d_j, \tilde{\mathbf{w}})$, $j = 1, 2, 3, 4$, are regular points of problem (1.3) when $d_1 \geqslant \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$. This, combined with Lemma 5.3(1), yields that problem (1.3) has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$ when $d_1 \geqslant \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$.

(3) For $0 < m_1 m_2 < 1$, if $a \geqslant \frac{2m_1 b}{1-m_1 m_2}$, then we have $m_1\tilde{v} \leqslant \tilde{u}$. By (4.7) and (4.8), we have $H(\mu_i) > 0$ for all $i \geqslant 0$. Thus, $\Gamma_p \cap \sum_{d_j}(d_j) = \emptyset$, $j = 1, 2, 3, 4$. Hence, by using Lemma 5.1, all of $(d_j, \tilde{\mathbf{w}})$ are regular points of problem (1.3). This, combined with Lemma 5.3(1), yields that problem (1.3) has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$. □

Lemma 5.4.

(1) *Suppose that $a \leqslant m_1 b$ and $d_4 \equiv 0$. There exist positive constants D_1^0 and D_2^0 such that when $d_1 > D_1^0$ and $d_2 > D_2^0$, problem* (1.3) *has no positive solution.*
(2) *Problem* (1.3) *has no non-constant positive solution bifurcating from* $\mathbf{0}$ *or* $\mathbf{u}^*$.
(3) *Problem* (1.3) *has no non-constant positive solution bifurcating from* $\mathbf{v}^*$ *if $a \neq m_1 b$.*

Proof. The proof process of Lemma 5.4(1) is similar to that of Lemma 5.2(1), so we here omit. Whereas, by Lemma 3.3, the positive solution $(u, v)^T$ of problem (1.3) satisfies $\min_{\overline{\Omega}} v > C^{-1}$. Furthermore, if $a \neq m_1 b$, then $\min_{\overline{\Omega}} u > C^{-1}$. Therefore, Lemma 5.3(2)–(3) holds. □

Remark 5.1. For $a = m_1 b$, since $\det\{\mathbf{G}_{\mathbf{w}}(\mathbf{v}^*)\} = 0$, $D_{\mathbf{w}}\mathbf{F}(\mathbf{v}^*)$ is not invertible. Therefore, our discussion is exclusive of this case.

6. The existence and bifurcation of non-constant positive solutions

By Lemmas 5.2–5.4, if problem (1.3) has non-constant positive solutions bifurcating from $\tilde{\mathbf{w}}$, then the following conditions hold:

(i) a satisfies: (1) $m_1 b < a < \frac{2m_1 b}{1-m_1 m_2}$ when $0 < m_1 m_2 < 1$, or (2) $a > m_1 b$ when $m_1 m_2 \geqslant 1$;
(ii) (d_1, d_2, d_3, d_4) satisfies that $0 < d_1 < \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, $d_2 > 0$, $d_3 \geqslant 0$ and $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$.

In the following, we first study the existence of non-constant positive solutions to problem (1.3).

Theorem 6.1. *Suppose that a satisfies the assumption of Remark* 3.2. *Let d_1, d_3 and d_4 be fixed and satisfy that $0 < d_1 < \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, $d_3 \geqslant 0$ and $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$, and let $D_2(d_1, d_3, d_4)$ be a positive number defined in Lemma* 4.2. *For $d_2 > D_2(d_1, d_3, d_4)$, if there exists some $n \geqslant 1$ such that $\mu_0 < \mu_1(\mathbf{d}) < \mu_1$, $\mu_2(\mathbf{d}) \in (\mu_n, \mu_{n+1})$, and the sum $\sigma_n = \sum_{i=1}^{n} \dim E(\mu_i)$ is odd, then problem* (1.3) *has non-constant positive solutions.*

Proof. Let D_2^0 and D_4^0 be as defined in Lemma 5.2(1), and let D_1, D_2, D_3 and D_4 be positive constants and satisfy that $D_1 < d_1$, $D_2 < \min\{D_2^0, D_2(d_1, d_3, d_4)\}$, $D_3 > d_3$ and $D_4 > d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}} \geqslant D_4^0$, respectively. For $t \in [0, 1]$, define $\mathbf{w} = (u, v)^T$,

$$
\begin{aligned}
&d_1(t) \equiv d_1, \qquad d_2(t) = 2D_2^0 + t\big(d_2 - 2D_2^0\big),\\
&d_3(t) \equiv d_3, \qquad d_4(t) = \frac{D_4^0}{2} + t\left(d_4 - \frac{D_4^0}{2}\right),\\
&\mathbf{\Phi}(t;\mathbf{w}) = \big(d_1(t)u\big(1 + d_3(t)v\big), d_2(t)v\big(1 + d_4(t)u\big)\big)^T,\\
&\mathbf{G}(\mathbf{w}) = \big(au - u^2 - m_1uv, bv - v^2 + m_2uv\big)^T,
\end{aligned}
$$

and consider the problem

$$
\begin{cases}
-\Delta\mathbf{\Phi}(t;\mathbf{w}) = \mathbf{G}(\mathbf{w}), & x \in \Omega,\\
\dfrac{\partial\mathbf{\Phi}(t;\mathbf{w})}{\partial\nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{6.1}
$$

Then $\mathbf{w}$ is a non-constant positive solution of problem (1.3) if and only if it is such a solution of problem (6.1) for $t = 1$. It is obvious that $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{v})^T$ is the unique constant positive solution of problem (6.1) for any $0 \leqslant t \leqslant 1$. As we observed in (4.2), for any $0 \leqslant t \leqslant 1$, $\mathbf{w}$ is a positive solution of problem (6.1) if and only if

$$
\begin{aligned}
\mathbf{F}(t;\mathbf{w}) &\triangleq \mathbf{w} - (\mathbf{I} - \Delta)^{-1}\big\{\big[\mathbf{\Phi}_{\mathbf{w}}(t;\mathbf{w})\big]^{-1}\big[\mathbf{G}(\mathbf{w}) + \nabla\mathbf{w}\mathbf{\Phi}_{\mathbf{ww}}(t;\mathbf{w})\nabla\mathbf{w}\big] + \mathbf{w}\big\}\\
&= 0, \quad \mathbf{w} \in \mathbf{X}^+.
\end{aligned}
\tag{6.2}
$$

It is obvious that $\mathbf{F}(1;\mathbf{w}) = \mathbf{F}(\mathbf{w})$. Lemma 5.2(1) shows that $\mathbf{F}(0;\mathbf{w}) = 0$ only has the positive solution $\tilde{\mathbf{w}}$ in $\mathbf{X}^+$. By a direct computation, we have

$$
D_{\mathbf{w}}\mathbf{F}(t;\tilde{\mathbf{w}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\big\{\big[\mathbf{\Phi}_{\mathbf{w}}(t;\tilde{\mathbf{w}})\big]^{-1}\mathbf{G}(\tilde{\mathbf{w}}) + \mathbf{I}\big\}.
$$

For $t = 1$, by making use of the condition of Theorem 6.1 and Lemma 4.3, we have $H(\mu_0) = H(0) > 0$, $H(\mu_i) < 0$ for $1 \leqslant i \leqslant n$, $H(\mu_i) > 0$ for $i > n$, where $H(\mu)$ is as defined in (4.6). Thus

$$
\sum_{H(\mu_i)<0} \dim\mathbf{E}(\mu_i) = \sum_{i=1}^{n} \dim\mathbf{E}(\mu_i) = \sigma_n
$$

which is odd. Thanks to Lemma 4.1, we have

$$
\operatorname{index}\big(\mathbf{F}(1;\cdot), \tilde{\mathbf{w}}\big) = (-1)^{\sigma_n} = -1.
$$

For $t = 0$, since $d_4(0) = \frac{D_4^0}{2} < \frac{1}{m_1\tilde{v} - \tilde{u}}$, by (4.7) and (4.8), we have $H(\mu_i) > 0$ for all $i \geqslant 0$. Therefore, by a similar argument, we have

$$
\operatorname{index}\big(\mathbf{F}(0;\cdot), \tilde{\mathbf{w}}\big) = (-1)^0 = 1.
$$

On other hand, by Lemma 3.3, there exists a positive constant C such that $\mathbf{F}(t;\mathbf{w}) \neq 0$ on $\partial\mathbf{B}(C)$ for all $0 \leqslant t \leqslant 1$. Therefore, by the homotopy invariance of the topological degree, we have

$$
\deg\big(\mathbf{F}(1;\cdot), \mathbf{0}, \mathbf{B}(C)\big) = \deg\big(\mathbf{F}(0;\cdot), \mathbf{0}, \mathbf{B}(C)\big).
$$

If $\mathbf{F}(1;\mathbf{w}) = 0$ has no non-constant positive solution on $\mathbf{B}(C)$, then

$$
\deg\big(\mathbf{F}(1;\cdot), \mathbf{0}, \mathbf{B}(C)\big) = \deg\big(\mathbf{F}(0;\cdot), \mathbf{0}, \mathbf{B}(C)\big) = \operatorname{index}\big(\mathbf{F}(0;\cdot), \tilde{\mathbf{w}}\big) = 1,
$$

which contradicts that $\deg(\mathbf{F}(1;\cdot), \mathbf{0}, \mathbf{B}(C)) = \operatorname{index}(\mathbf{F}(1;\cdot), \tilde{\mathbf{w}}) = -1$. Hence, problem (1.3) has non-constant positive solutions.

Further, by (4.12), if d_1, d_3 and d_4 satisfy that

$$\lim_{d_2\to\infty} \mu_2(\mathbf{d}) = \frac{\tilde{u}[d_4(m_1\tilde{v} - \tilde{u}) - 1]}{d_1(1 + d_3\tilde{v} + d_4\tilde{u})} \in (\mu_n, \mu_{n+1}),$$

then let $D_2(d_1, d_3, d_4)$ be enlarged such that $0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) \in (\mu_n, \mu_{n+1})$ for all $d_2 > D_2(d_1, d_3, d_4)$. Thus, in this case, problem (1.3) always has non-constant positive solutions for all $d_2 > D_2(d_1, d_3, d_4)$. Furthermore, by a similar argument as that in (3.5), we have $(1 + d_4u)v = C$ as $d_2 \to \infty$, and u satisfies

$$\begin{cases} -d_1\Delta\left[u\left(1 + \dfrac{d_3C}{1 + d_4u}\right)\right] = au - u^2 - \dfrac{m_1Cu}{1 + d_4u}, \\ \displaystyle\int_\Omega \frac{C}{1 + d_4u}\left(b - \frac{C}{1 + d_4u} + m_2u\right) dx = 0, \end{cases} \tag{6.3}$$

where C is a positive constant, the second equation of (6.3) is obtained by integrating the second equation of problem (1.3) and letting the limit, the first equation of (6.3) is the limit of the first equation of problem (1.3), and here let $\frac{C}{1+d_3u}$ take the place of v. □

By a similar argument as Theorem 6.1, we have Theorems 6.2 and 6.3 in the general case.

Theorem 6.2. *Suppose that a satisfies the assumption of Remark* 3.2. *Let d_1, d_2 and d_3 be fixed and satisfy that $0 < d_1 < \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, $d_2 > 0$ and $d_3 \geqslant 0$, and let $D_4(d_1, d_2, d_3)$ a positive number defined in Lemma* 4.2 *and satisfy that $D_4(d_1, d_2, d_3) \geqslant \frac{1}{m_1\tilde{v}-\tilde{u}}$. For $d_4 > D_4(d_1, d_2, d_3)$, if there exists some $n \geqslant 1$ such that $\mu_0 < \mu_1(\mathbf{d}) < \mu_1$, $\mu_2(\mathbf{d}) \in (\mu_n, \mu_{n+1})$, and the sum $\sigma_n = \sum_{i=1}^n \dim E(\mu_i)$ is odd, then problem* (1.3) *has non-constant positive solutions.*

Remark 6.1. Suppose that the sum $\sum_{i=1}^n \dim E(\mu_i)$ is odd.

(1) Let d_1, d_3 and d_4 be fixed and satisfy $0 < d_1 < \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, $d_3 \geqslant 0$, $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$ and $\frac{\tilde{u}[d_4(m_1\tilde{v}-\tilde{u})-1]}{d_1(1+d_3\tilde{v}+d_4\tilde{u})} \in (\mu_n, \mu_{n+1})$, then there exists a positive constant $D_2(d_1, d_3, d_4)$ such that $\mu_0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) \in (\mu_n, \mu_{n+1})$ for all $d_2 > D_2(d_1, d_3, d_4)$.

(2) Let d_1, d_2 and d_3 be fixed and satisfy $0 < d_1 < \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, $d_2 > 0$, $d_3 \geqslant 0$ and $\frac{m_1\tilde{v}-\tilde{u}}{d_1} \in (\mu_n, \mu_{n+1})$, then there exists a positive constant $D_4(d_1, d_2, d_3)$ satisfying $D_4(d_1, d_2, d_3) > \frac{1}{m_1\tilde{v}-\tilde{u}}$ such that $\mu_0 < \mu_1(\mathbf{d}) < \mu_1$ and $\mu_2(\mathbf{d}) \in (\mu_n, \mu_{n+1})$ for all $d_4 > D_4(d_1, d_2, d_3)$.

Theorem 6.3. *Suppose that a satisfies the assumption of Remark* 3.2. *Let d_1, d_2, d_3 and d_4 satisfy that $0 < d_1 < \frac{m_1\tilde{v}-\tilde{u}}{\mu_1}$, $d_2 > 0$, $d_3 \geqslant 0$ and $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$. If there exist two integers $n_2 > n_1 \geqslant 0$ such that $\mu_1(\mathbf{d}) \in (\mu_{n_1}, \mu_{n_1+1})$, $\mu_2(\mathbf{d}) \in (\mu_{n_2}, \mu_{n_2+1})$, and the sum $\sigma_n = \sum_{i=n_1+1}^{n_2} \dim E(\mu_i)$ is odd, then problem* (1.3) *has non-constant positive solutions.*

Next, we consider the bifurcation of non-constant positive solutions with respect to d_1. Suppose that a satisfies the assumption of Remark 3.2. Let d_2, d_3, d_4 be fixed and satisfy that $d_2 > 0$, $d_3 \geqslant 0$, $d_4 > \frac{1}{m_1\tilde{v}-\tilde{u}}$, and one of d_2 and d_4 is large enough, then we have the following theorem. Its proof is a similar treatment in [13].

Theorem 6.4.

(1) *If* $\Gamma_p \cap \mathbf{S}_{d_1}(\tilde{d}_1) = \emptyset$, *then* $(\tilde{d}_1, \tilde{\mathbf{w}})$ *is a regular point of problem* (1.3).
(2) *Suppose that* $\Gamma_p \cap \mathbf{S}_{d_1}(\tilde{d}_1) \neq \emptyset$ *and the positive roots of* $H(\mu) = 0$ *are all simple. If* $\sum_{\mu_i \in \mathbf{S}_{d_1}(\tilde{d}_1)} \dim E(\mu_i)$ *is odd, then* $(\tilde{d}_1, \tilde{\mathbf{w}})$ *is a bifurcation point of problem* (1.3). *In this case, there exists an interval* $(\underline{d}_1, \bar{d}_1) \in \mathbf{R}^+$, *where*
 (i) $\tilde{d}_1 = \underline{d}_1 < \bar{d}_1 < +\infty$, $\Gamma_p \cap \mathbf{S}_{d_1}(\bar{d}_1) \neq \emptyset$, *or*
 (ii) $0 < \underline{d}_1 < \bar{d}_1 = \tilde{d}_1$, $\Gamma_p \cap \mathbf{S}_{d_1}(\underline{d}_1) \neq \emptyset$,
 such that, for every $d_1 \in (\underline{d}_1, \bar{d}_1)$, *problem* (1.3) *admits a non-constant positive solution.*

Remark 6.2. Let $k_1 = k_2 = 1$. If we directly consider the positive steady states of system (1.1), then the assumption of Remark 3.2 will be replaced by $m_1 < r_1 < 2m_1 + \frac{m_1 m_2}{r_2}$. By Lemmas 5.3 and 5.4, we know that, when $r_1 \geqslant 2m_1 + \frac{m_1 m_2}{r_2}$ (or $r_1 < m_1$), the stationary problem of system (1.1) has no non-constant positive solution bifurcating from $\tilde{\mathbf{w}}$ (or $\mathbf{v}^*$).

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